Dynamics of the Space Shuttle during Entry into Earth's Atmosphere

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The problem of predicting the longitudinal dynamics of a shuttle vehicle during entry into the Earth's atmosphere is investigated in this paper. Subject to certain conditions, the equations of motion are transformed by a change of variables corresponding to the number of scale lengths traversed by the vehicle along the trajectory of the center of mass. Asymptotic solutions to the transformed equations of motion are developed by means of the technique of generalized multiple scales by introducing nonlinear "scale functions." Rapid and slow parts of the time-varying dynamics of the vehicle are systematically separated in this approach. By means of this theory, the longitudinal dynamics of the SSV 049 vehicle about a prescribed optimal trajectory are predicted. The trajectory is optimal in that it minimizes the weight of the thermal protection system. Solutions are developed to describe the unforced (free) dynamics as well as the response to a time-varying input. Comparison of the analytical and numerical solutions in both cases shows excellent agreement and the difference is hardly noticeable for engineering purposes. Further, an analysis of the errors is presented leading to explicit *analytical* bounds (which are strict and sharp) on the errors of the approximation. The advantages of our approach are: 1) analytical approximations to the dynamics of the shuttle vehicle are obtained in terms of *simply calculable* elementary transcendental functions; 2) the separation of the fast and slow dynamics is displayed explicitly in the general case, leading to insight into the nature of the shuttle entry dynamics; 3) the approach is very general and is widely applicable to any type of vehicle and any entry trajectory into the atmosphere of the Earth or any other planet; 4) strict and sharp bounds on the errors of the approximation are known at every point of the trajectory.

Nomenclature

a, b = initial and final points of an interval
C = slowly varying amplitude, as in Eq. (18)
C_D = drag coefficient
C_L = lift coefficient
C_m = pitching moment coefficient
$C_{D_\alpha}, C_{L_\alpha}, C_{m_\alpha}, C_{m_{\dot\alpha}}, C_{m_q}$ = stability derivatives
f = inhomogeneous or forcing term in the unified Eq. (8)
g = acceleration due to gravity
h = altitude
i = $+(-1)^{1/2}$
I_x, I_y, I_z = principal moments of inertia
$\bar{k}$ = clock function
L = characteristic length
m = mass of vehicle
q = angular velocity in pitch relative to the Earth
r = radial distance from the center of Earth
s = dummy variable of integration
S = reference area
t = time
V = speed along trajectory
α_0 = initial angle of attack
$\bar{\alpha}$ = angle of attack
α = $\bar{\alpha}-\alpha_0$ = perturbation angle of attack
γ = flight-path angle
δ = nondimensional mass of the atmosphere
ε = measure of slowness of variation of the coefficients, Eq. (8) and Eq. (11)
θ = pitch angle
ν = ratio of moments of inertia
ξ = universal independent variable
ρ = air mass density
σ = inverse nondimensional pitching moment of inertia
τ_0, τ_1 = new independent variables due to extension
$\omega_1, \omega_0, \Omega_1, \Omega_0$ = coefficients in the corresponding equation

Subscripts

s = slowly varying quantity
f = rapidly varying quantity
r = real part
i = imaginary part
p = particular solution
0 = initial value
f = final value in Eq. (11)
f = forced solution in Eq. (26)

Received April 1, 1974; revision received September 9, 1974. Support of this investigation by the NASA Lyndon B. Johnson Space Center under Contract NAS 9-4065 is gratefully acknowledged.

Index categories: Entry Vehicle Dynamics and Control; Aircraft Handling, Stability, and Control; LV/M Dynamics and Control.

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I. Introduction

THIS paper is concerned with the development of an analytical, asymptotic representation of the dynamics, in the plane of symmetry, of a space shuttle vehicle during entry into the Earth's atmosphere. The solution is obtained by separating the rapid and slow parts of the total dynamics, by the generalized multiple scales method.

The problem of analyzing the dynamics of a space vehicle during entry into the atmosphere has generated considerable research interest. The equations of motion are nonlinear and no exact analytical solutions have been found. Many approximate solutions have been proposed, based on intuitive assumptions on the nature of the forces and moments affecting the motion. In deriving the approximations, usually the gravity force is neglected in comparison with the aerodynamic force when considering short-period oscillations.[1-6] Such an analysis is necessarily valid mainly in the ballistic part of the entry which has a large deceleration. Under such conditions Allen[2] showed that the dynamics can be described in terms of Bessel functions. Loh[4] developed an analytical approximation valid over fairly wide conditions. Citron and Meir[6] studied approximations for density, altitude, and flight-path angle which they obtained through a series expansion in a variable depending on velocity. On the other hand, the dynamics of vehicles at very small flight-path angles has been studied by Etkin[8] and others. It can be shown that a comprehensive approach can be developed, valid for a general class of trajectories. Vinh and Laitone[9] derive a unified linear differential equation describing angle-of-attack variations which is valid for all entry trajectories. This requires the choice of the distance traversed by the center of mass of the vehicle as the independent variable. However, this equation has variable coefficients and cannot be solved exactly in general. Vinh and Laitone derive approximate solutions for two specific entry trajectories—straight line entry and those with shallow flight-path angle. In the former case, the equation becomes the confluent hypergeometric (or Kummer's) equation which simplifies to Bessel's equation of zeroth-order for a ballistic missile. For a shallow entry the dynamics were shown to be described by an inhomogeneous, damped Mathieu equation.[9]

Since the above two cases constitute two distinct types of entry which are analytically tractable, it would seem to be very useful to develop analytical representation of the motions in the general case. Exact solution of the general equation is rendered impossible because of the variable coefficients. In this paper, asymptotic solutions are developed for the general angle-of-attack variations, when the coefficients vary relatively slowly with respect to the oscillations. The concept of slow variation can be rigorously defined and is discussed in the paper. By employing fast and slow scales, which depend nonlinearly on the distance traveled along the flight path, analytical asymptotic solutions are derived for the angle-of-attack variations in a separable form. The advantage is that these solutions are valid for a very large class of entry trajectories and vehicles. These solutions are shown to be accurate and, because they are analytical, they can be used for a variety of stability and control investigations.

II. Equations of Motion

In the following equations describing the motion of the vehicle in the plane of symmetry, it is assumed that the vehicle experiences lift but does not roll and that the planet is spherical. The axis system through the center of mass of the vehicle is such that the x-axis is always tangential to the instantaneous flight path. The equations of motion for arbitrary flight path angles and zero thrust are given by[7,9]:

$$\dot{V} = -\rho S C_D V^2/(2m) - g\sin\gamma \tag{1}$$

$$V\dot{\gamma} = \frac{\rho S C_L V^2}{2m} - \left(g - \frac{V^2}{r}\right)\cos\gamma \tag{2}$$

$$\dot{q} = \frac{\rho S L C_m V^2}{2I_y} - \frac{3g}{2r}\left(\frac{I_x - I_z}{I_y}\right)\sin 2\theta \tag{3}$$

and by the kinematic relations

$$\dot{\theta} = q + (V/r)\cos\gamma \tag{4}$$

$$\dot{r} = V\sin\gamma \tag{5}$$

$$\theta = \gamma + \bar{\alpha} \tag{6}$$

where the dot denotes differentiation with respect to time. We assume that the slope of the lift curve is approximately independent of the flight speed and Mach number in high supersonic flight and linearize the aerodynamic coefficients about the nominal trajectory. Following,[9] elimination of θ and V from Eqs. (1–6) and change of the independent variable from t to ξ according to

$$L\dot{\xi} = V(t) \tag{7}$$

leads to the following equation for perturbation angle-of-attack α after linearizing the aerodynamic coefficients.

$$\alpha'' + \omega_1(\xi)\alpha' + \omega_0(\xi)\alpha = f(\xi) \tag{8}$$

where

$$\omega_1(\xi) = \delta\left[C_{L_\alpha} - \sigma(C_{m_\alpha} + C_{m_q})\right] + \frac{V'}{V}$$

$$\omega_0(\xi) = -\delta\left(\sigma C_{m_\alpha} + \frac{gL}{V^2}C_{D_\alpha}\cos\gamma\right) + \delta' C_{L_{\dot{\alpha}}} + \delta\frac{V'}{V}C_{L_\alpha} - \delta^2\left[C_{L_\alpha}(\sigma C_{m_q} + C_{D_0}) + C_{L_0}C_{D_\alpha}\right] + \frac{3L}{r}\left(\frac{gL}{V^2}\right)\nu\cos 2(\gamma + \alpha_0)$$

$$f(\xi) = \delta\left(\frac{gL}{V^2}\right)\left[C_{D_0} - \sigma C_{m_q}\left(1 - \frac{V^2}{gr}\right)\right]\cos\gamma - \delta' C_{L_0} + \delta^2 C_{L_0}(C_{D_0} + \sigma C_{m_q}) - \left(\frac{gL}{V^2}\right)\left[\left(\frac{3L}{r} - \frac{gL}{V^2}\right)\sin 2\gamma + \frac{3L}{2r}\nu\sin 2(\gamma + \alpha_0)\right] \tag{9}$$

and

$$\delta = \frac{\delta S L}{2m}, \quad \nu = \frac{I_x - I_z}{I_y}, \quad \sigma = \frac{mL^2}{I_y} \tag{10}$$

The primes denote differentiation with respect to ξ. The terms ω_1, ω_0, and f are solely functions of the independent variables ξ and can be determined explicitly if the trajectory flown by the center of mass is known. This implies that the effect of the angle-of-attack perturbations on the trajectory itself is negligible. It is, in general, impossible to integrate Eq. (8) exactly. Vinh and Laitone show that for two specific entry trajectories Eq. (8) reduces to well-known equations in mathematical physics. For a ballistic entry at steep angles Eq. (8) reduces to the Kummer's confluent hypergeometric equation, which can be further simplified into Bessel's equations of zeroth-order. For a shallow gliding entry, however, the equation becomes of the damped Mathieu form with periodic forcing terms. From these two well-known forms, asymptotic solutions can be developed, providing an analytical feel for the system dynamics. In particular, Vinh and Laitone discuss only the solutions of the homogeneous equations as they are primarily interested in stability.

An actual trajectory does not, in general, correspond exactly to the two types considered, and an analytical description of the motion is thus rendered more difficult. Nonetheless, it is possible to develop useful approximations to the general problem by considering the dynamics to occur faster than the change in the coefficients. We are thus considering a more general class of trajectories. We only require that the trajectories be such that the coefficients of Eq. (8) be slowly varying functions of ξ. It turns out that this is a valid approach to the problem being studied. Asymptotic analytical solutions can be developed, as follows, in terms of elementary functions as opposed to special functions such as those of Bessel, Kummer, or Mathieu.

III. Development of Solution

As we have noted, Eq. (8) cannot, in general, be solved exactly. Only for particular variation of the coefficients can the equation be solved in terms of special mathematical functions, which are non-elementary. Even with these exact solutions, greater insight is obtained by studying their asymptotic representations than by the exact forms, which are available only as tabulated values. This is illustrated, for example, in the paper by Vinh and Laitone. For the problem considered in this paper, for the case of ballistic trajectory along a straight line, Vinh and Laitone study the classical asymptotic solutions to be confluent hypergeometric equation. For the case of the shallow gliding entry represented by an inhomogeneous, damped Mathieu equation they develop approximate solutions by the method of Krylov and Bogoliubov. In contrast to solving the special equations for the two particular trajectories, it is possible to develop approximate solutions in the general case, by methods such as WKBJ and the method of averaging. In this paper approximate solutions will be developed for the general equation by the asymptotic method of multiple scales. Indeed, for the present problem, this approximation is equivalent to the WKBJ solution. However, we feel that the dynamics are rendered more transparent by the multiple scales approach because of the systematic separation of fast and slow motions.

Although the method has been developed relatively recently, it is rapidly becoming well known.[10–13] In view of this trend, the method will not be discussed in detail in this paper. For further details the reader can consult Refs. 10–13. Briefly, however, the method enables us to develop asymptotic solutions to dynamic problems by separating the fast and slow parts of the dynamics. In order to achieve such a separation the independent variable is extended into a space of higher dimension by means of (in general, nonlinear) scale functions. The resulting equations are solved asymptotically and the solutions are then restricted to the original problem variables.

Experience with the entry trajectories of missiles and the space shuttle suggests that the coefficients of Eq. (8) can be realistically considered to be slowly varying. For example, in traversing the Earth's atmosphere from a fringe altitude of about 400,000 ft, the variations in the coefficients are primarily due to changes in density, velocity, and the aerodynamic force and moment parameters along the three-dimensional entry trajectory. Such variations are slow compared to the time constant of the motion of the vehicle. Mathematically, it is tantamount to saying that the coefficients of Eq. (8) vary on a new slow variable $\bar{\xi} = \varepsilon\xi$, where ε is a small, positive parameter, being a measure of the ratio of the time constants of the dynamic motion and the coefficient variation. We will develop asymptotic solutions as this separation becomes greater, or equivalently, as $\varepsilon \to 0$. Equation (8) can now be written as:

$$\varepsilon^2\alpha'' + \varepsilon\omega_1(\bar{\xi})\alpha' + \omega_0(\bar{\xi})\alpha = f; \quad \bar{\xi} \in (\xi_0, \xi_f) \tag{11}$$

where the primes denote differentiation with respect to $\bar{\xi}$ and the coefficients are appropriately given by Eq. (9). In order to develop asymptotic solutions to Eq. (11), we will invoke the concept of multiple scales. The argument is as follows. Solution of the inhomogeneous Eq. (11) can be constructed if the solution to the homogeneous equation is known. We propose to develop asymptotic solutions to the forced system Eq. (11) by first constructing approximate analytical solutions to the homogeneous problem by the generalized multiple scales method.[10,11] The method of variation of parameters then leads to the solution of Eq. (11).

In order to obtain approximate solutions to the homogeneous problem, we extend the variables as follows.

$$\bar{\xi} \to \{\tau_0, \tau_1\}; \quad \tau_0 = \bar{\xi}, \quad \tau_1 = \int \frac{\bar{k}}{\varepsilon}(\bar{\xi})\,d\bar{\xi}$$
$$\alpha(\bar{\xi}, \varepsilon) \to \alpha(\tau_0, \tau_1) \tag{12}$$

In the light of multiple scales theory,[10–13] the zeroth- and first-order extended perturbation equations are:

$$\bar{k}^2 \frac{\partial^2 \alpha}{\partial \tau_1^2} + \omega_1 \bar{k} \frac{\partial \alpha}{\partial \tau_1} + \omega_0 \alpha = 0 \tag{13}$$

$$\dot{\bar{k}} \frac{\partial \alpha}{\partial \tau_1} + 2\bar{k} \frac{\partial^2 \alpha}{\partial \tau_0 \partial \tau_1} + \omega_1 \frac{\partial \alpha}{\partial \tau_0} = 0 \tag{14}$$

We seek solutions to Eq. (13) in the form:

$$\alpha(\tau_0, \tau_1) = \alpha_s(\tau_0)\alpha_f(\tau_1) \tag{15}$$

Upon substituting Eq. (15) into Eqs. (13) and (14) and simplifying, we obtain:

$$\alpha_f(\tau_1) = \exp(\tau_1) \tag{16}$$

where the clock $\bar{k}$ is given by

$$\bar{k}^2 + \omega_1 \bar{k} + \omega_0 = 0 \tag{17}$$

The slow solution α_s is given by:

$$\alpha_s(\tau_0) = (\omega_1{}^2 - 4\omega_0)^{-1/4} C(\tau_0) \tag{18}$$

where $C(\tau_0)$ is a slowly varying quantity given by:

$$\frac{\partial}{\partial \tau_0}(\ln C) = \frac{\partial}{\partial \tau_0}[\ln(\omega_1{}^2 - 4\omega_0)^{1/4}] \tag{19}$$

In this paper C is considered a pure constant (justified if ω_1 and ω_0 vary slowly enough). In oscillatory phenomena the clock $\bar{k}(\bar{\xi})$ is necessarily a complex quantity. Therefore, we write

$$\bar{k}(\tau_0) = k_r(\tau_0) + i k_i(\tau_0) \tag{20}$$

and denote the real and imaginary parts. Upon restricting the extended solution of zeroth-order, i.e., the fast scale solution, along the trajectories Eq. (12), we obtain the first approximation to the general solution of Eq. (11) as:

$$\hat{\alpha}(\bar{\xi}) = \alpha_f(\tau_1)|_{\tau_1(\bar{\xi})} = C_1 \hat{\alpha}_1(\bar{\xi}) + C_2 \hat{\alpha}_2(\bar{\xi}) \tag{21}$$

where

$$\hat{\alpha}_1(\bar{\xi}) = \exp\left(\int_{\xi_0}^{\bar{\xi}} k_r(\tau_0)\,d\tau_0\right) \sin\left(\int_{\xi_0}^{\bar{\xi}} k_i(\tau_0)\,d\tau_0\right) \tag{22}$$

$$\hat{\alpha}_2(\bar{\xi}) = \exp\left(\int_{\xi_0}^{\bar{\xi}} k_r(\tau_0)\,d\tau_0\right) \cos\left(\int_{\xi_0}^{\bar{\xi}} k_i(\tau_0)\,d\tau_0\right) \tag{23}$$

C_1, C_2 are arbitrary constants. As these are fast scale solutions, they primarily describe variations in frequency and phase of the solution. A more accurate solution can be obtained by including the slower behavior as well, i.e., by including the first-order solution. A second approximation is written as

$$\tilde{\alpha}(\bar{\xi}) = \alpha_s(\tau_0)|_{\tau_0(\bar{\xi})} \alpha_f(\tau_1)|_{\tau_1(\bar{\xi})} \tag{24}$$

i.e.,

$$\tilde{\alpha}(\bar{\xi}) = D^{-1/4}(\bar{\xi})\hat{\alpha}(\bar{\xi}) \tag{25}$$

where $D(\xi)$ is the absolute value of the discriminant of Eq. (17). Each of the solutions $\hat{\alpha}(\bar{\xi})$ and $\tilde{\alpha}(\bar{\xi})$ leads to the particular solution α_p of the inhomogeneous Eq. (11) by the method of variation of parameters. The approximate general solution is given by:

$$\alpha_f{}^*(\xi) = C_1 \alpha_1{}^*(\xi) + C_2 \alpha_2{}^*(\xi) + \alpha_p{}^*(\xi) \tag{26}$$

The fast only or fast and slow solutions $\hat{\alpha}$ or $\tilde{\alpha}$ solutions are substituted for α^* in Eq. (26) to generate the corresponding approximate solutions. It must be noted that in developing the above approximations to the forced system Eq. (11) we have considered the nonresonant case, i.e., f is *not* of the form $\exp[n(\bar{\xi})]$ in which $n'(\xi) = \bar{k}(\bar{\xi})$.

IV. Application

The angle-of-attack oscillations of the space shuttle are now studied by the multiple scales theory. The system considered is the space shuttle 049 vehicle and the dynamics are studied about an entry trajectory designed to minimize the thermal-protection-system (TPS) weight.[14] Figure 1 shows the variation of the trajectory variables angle-of-attack α_0 velocity V, altitude h, and flight path angle γ as a function of the new variable ξ, the nondimensional distance along the trajectory. Coefficients of the governing equations, after linearization about the prescribed trajectory, are now functions of the nominal angle-of-attack α_0. The variation of ξ with respect to real time (Fig. 1) is nonlinear. Upon substitution of the trajectory information into Eq. (9), solutions of the clock characteristic Eq. (17) showing the clock variations are shown in Fig. 2. The approximate solutions derived in the last section are employed directly to study the longitudinal motion of the shuttle vehicle. For purposes of comparison solutions are obtained for the initial conditions;

$$\tilde{\alpha}(\xi_0) = 0, \quad \tilde{\alpha}'(\xi_0) = k_i(\xi_0) \tag{27}$$

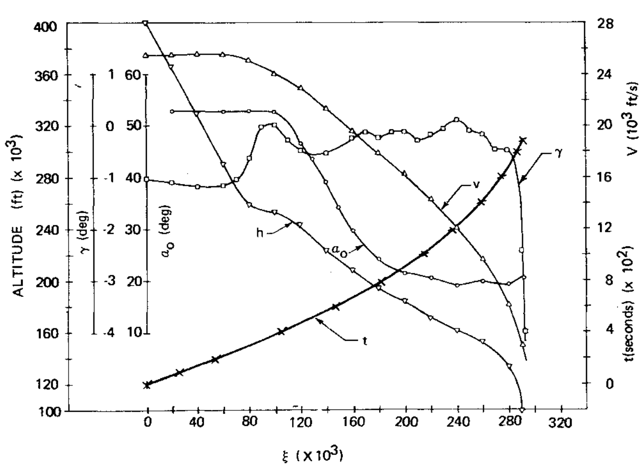

Fig. 1 Trajectory characteristics.

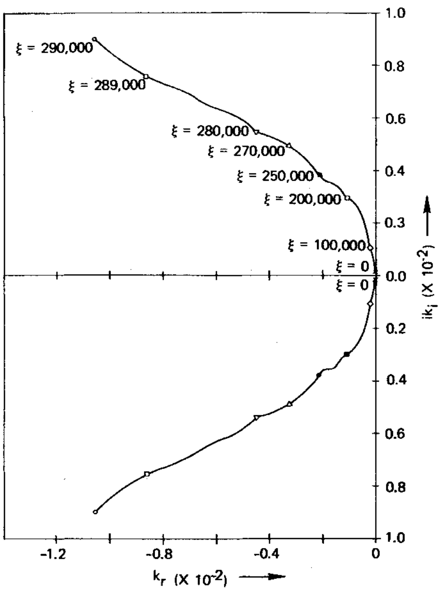

Fig. 2 Variation of complex clock.

for the sine-like solutions and

$$\tilde{\alpha}(\xi_0) = 1, \quad \tilde{\alpha}'(\xi_0) = k_r(\xi_0) \tag{28}$$

for the cosine-like solution. The approximate solutions are shown in Figs. 3 and 4.

The homogeneous solution, which represents the transient part, represent the angle-of-attack oscillation which increases in frequency as the shuttle travels deeper into the atmosphere of increasing density. The fast-scale solution, which represents the rapid motions, describes the variable frequency very well, as indicated by the zero-crossings of the fast-scale solution and the exact solution obtained by numerical integration. However, the variations in amplitude, which occur on a slower scale, are naturally in some error as predicted by the fast-scale solution, which primarily predicts the frequency variation. The approximation can be improved by including the slower effects as well, i.e., by a combination of the slow and fast solutions. Thus, the slow-scale solution corrects the amplitude error precisely at the required magnitudes at the proper times, without altering the frequency (Figs. 3 and 4). An interesting comparison is that of the "frozen" solution which is a consequence of "freezing" the coefficients at some time and considering a constant coefficient analysis. This approach is common in the engineering analysis of time-varying systems. Such a solution frozen at initial re-entry is shown in Figs. 3 and 4. Clearly it misrepresents the dynamics as early as during second quarter-cycle of the oscillation. In contrast the multiple scales solution provides a uniform, accurate description of the phenomenon. Even in response to the input, f, our approximation faithfully represents the true behavior uniformly throughout the entry region (Figs. 5a and 5b). We will now proceed to discuss the errors of the approximation.

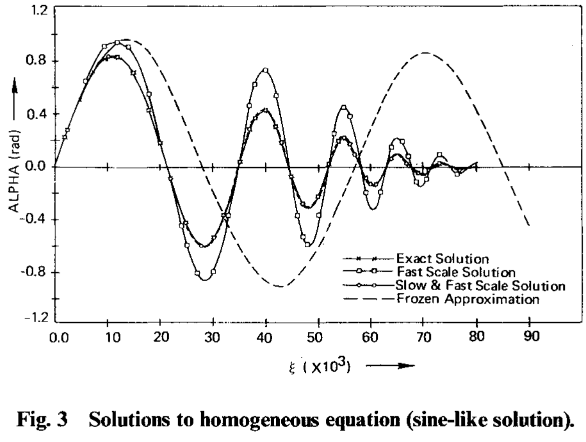

Fig. 3 Solutions to homogeneous equation (sine-like solution).

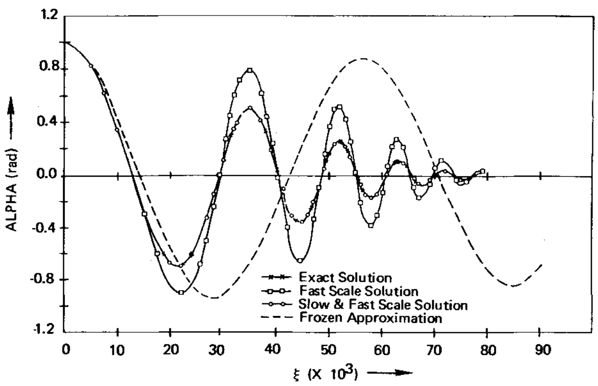

Fig. 4 Solutions to homogeneous equation (cosine-like solution).

V. Error Analysis

An important aspect of any approximation is the magnitude of errors associated with it. The problem of a priori prediction of errors is by no means simple, as it is tantamount to solving the original problem exactly. In the case of time-varying systems it is known that, in general, they cannot be solved exactly in a finite number elementary operations. In this apparent impasse of determining the errors without being able to solve the equations exactly there is a way out, in particular for the problem at hand. We can develop upper bounds on the errors between the exact and approximate solutions and these bounds are strict and sharp.

In this paper we will only be concerned with the errors of the unforced system dynamics, as predicted by the multiple scales theory. These describe the errors in the transient part of a forced solution. The errors in the forced response can be determined by quadratures over the input and the unforced solution.

The starting point of such an analysis is the following error, Theorem 10, which is based on Olver's work[15] on error bounds.

Theorem 10

The differential equation

$$\varepsilon^2 y'' + \varepsilon\Omega_1(t,\varepsilon)y' + \Omega_0(t,\varepsilon)y = 0; \quad t\in(a,b)$$

has conjugate solutions y and y^* such that

$$y(t,\varepsilon) = \tilde{y}(t,\varepsilon) + E$$

where

$$|E| \leq D^{-1/4}\exp\left(-\int_a^t \frac{\Omega_1}{2\varepsilon}dt + \varepsilon\left|\int_c^t \left|m(t,\varepsilon)\right|\left|dt\right|\right|\right) - 1$$

$$a < c < b, \quad 0 < \varepsilon \ll 1$$

$\tilde{y}(t,\varepsilon)$ is the multiple scales solution to first-order and D is the absolute value of the discriminant of the clock equation

$$\bar{k}^2 + \Omega_1\bar{k} + \Omega_0 = 0$$

and

$$m(t,\varepsilon) = D^{-1/4}\frac{d^2}{dt^2}(D^{-1/4})$$

The interval (a,b) and the value of c may be infinite provided that the integrals converge. We note that similar bounds can be given for the dy/dt also.

This theorem is valid for the case of oscillatory solutions and in a region free from turning points. The longitudinal motion of the shuttle satisfies these conditions and therefore the errors of the multiple scales solution Eq. (25) can be bound by means of the above theorem. Indeed, a more accurate theorem on the approximate solutions is given in Refs. 10 and 15.

Application of the theorem to the shuttle problem requires the substitution $\omega_1 = \Omega_1/\varepsilon$ and $\omega_0 = \Omega_0/\varepsilon^2$. The results on the errors would then be valid. The variation of the errors for the shuttle entry as predicted by Theorem 1 is shown in Fig. 6. It is seen that the errors are indeed small and that the multiple scales

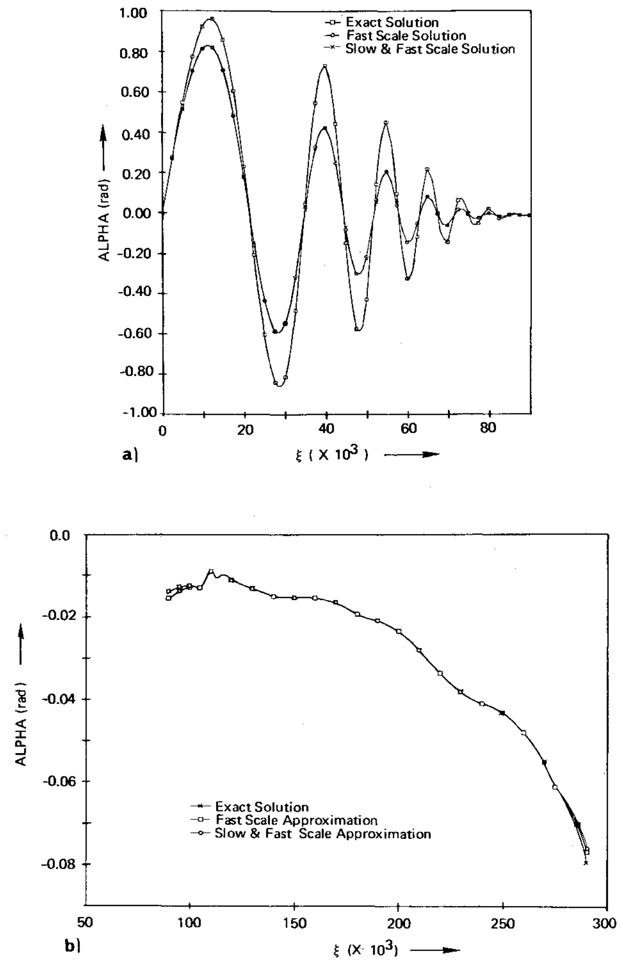

Fig. 5 Response to f.

approximation represents the angle-of-attack of oscillations extremely well. Further, the analytical prediction of errors is seen to be rigorous and effective. The frequency of the error propagation as predicted analytically is highly accurate and the magnitudes of the errors are indeed bound by the analytical limits. For the general slowly time-varying systems Ref. 10 presents a second theorem which yields stricter bounds on the errors. Indeed it can be shown that the error bounds stated here constitute in form the leading contribution to those in this second theorem.

From the error theorem it is seen that the system is slowly varying if $\int_\varepsilon^t p^{-1/4}(p^{-1/4})'' dt$ is small where $p(t,\varepsilon) = \Omega_0 - (\Omega_1{}^2/4) - (\varepsilon\dot{\Omega}_1/2)$. Since the integrand is defined to be positive the error is small if the integrand is small. This line of analysis can be pursued and leads to a simple criterion that the coefficients are slowly varying if $p'/p \ll 1$ provided that $p^{-1/4}$ is at most of order 1. This is consistent, for if $p'/p \sim \lambda \ll 1$, then we are led to $p = O(1)$. In any case, in the conditions we have considered, we need only the weaker form that the above integral is $O(1)$. For any finite t, however large, our results are valid and the error is small, as $\varepsilon \to 0$. More rigorously the integral form must be used.

VI. Summary and Conclusions

In this study we have investigated the angle-of-attack oscillations of a shuttle-type vehicle during entry into the Earth's atmosphere. The motion is studied with respect to a typical entry trajectory which is designed to minimize the weight of the thermal protection system. Analytical asymptotic solutions have been developed for the longitudinal motion of the space shuttle during entry into the Earth's atmosphere. The equations are first

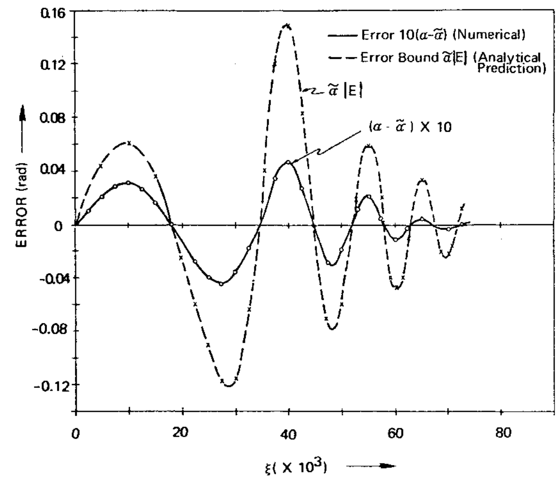

Fig. 6 Comparison of errors.

transformed by choosing the scaled distance along the trajectory as the new independent variable. The method of generalized multiple scales is now invoked, and approximate solutions are then developed, separating the rapid and slow parts of the solution. The multiple scales solutions are compared with numerical integrals of the equations. It is seen that the fast solution predicts frequency variations very well, but has a small error in amplitude. This is corrected by including the slow solution also. Thus, the approximation to first order represents the true solution extremely well. Further, it can be seen that the response to the input f can be accurately represented by considering only the fast-scale homogeneous solutions. Inclusion of the slow part of the homogeneous solution primarily results in a more accurate description of the transient part of the forced response.

Subject to the conditions stated our approach is valid for: 1) any type of vehicle, 2) any type of entry trajectory, 3) into the atmosphere of the Earth or any other planet. Our solutions exhibit the separation of the slow and rapid aspects of the dynamics. In order to do so the scales are necessarily nonlinear. Further, the error theorems provide strict and sharp bounds on the errors of the approximation. We have thus developed a criterion of rigorously determining "slow variation" of a time-varying shuttle dynamics, which is often treated intuitively. Exact analytical solutions in the general case being impossible, we have developed asymptotic analytical approximations to very good accuracy in the general slowly varying case.

In addition, we have been able to bound the errors of the approximation. An error theorem had been stated (proved in Ref. 10) for general slow coefficient variations depending on different entry trajectory. Having such an analytical solution,, new areas of analysis are opened up, such as multiparameter sensitivities, control and stability.

In contrast, it is useful to consider the Vinh and Laitone approach. Their analysis is interesting and clever. However, after deriving the unified equation, they obtain asymptotic solutions only in two special cases—1) steep, straight line trajectory and 2) shallow gliding entry. For case 1) they use asymptotic behavior of the confluent hypergeometric functions and Bessel functions of large order. Case 2 is solved by the Krylov-Bogoliubov technique. An actual trajectory will be somewhat between the two extremes of Cases 1 and 2. Although our solutions remind us of Case 1, no specific assumptions required for Case 1 (such as $gL/V^2 \ll 1$) have been made in our analysis. Besides, we do not assume $C_{L_0} = C_{D_\alpha} = 0$ as they have. A comparison of the difference between our approach and that of Vinh and Laitone is given in Table 1.

Finally we note that this mathematical approach can be used to study the dynamics of other time-varying systems, such as missiles, VTOL and aircraft over large flight envelopes.

Table 1 A comparison

Vinh-Laitone approach	Present approach
1) Assume $C_{L_0} = C_{D_\alpha} = 0$ even at high α	Even at high α ($=50°$) $C_{L_0} \neq 0$, $C_{D_\alpha} \neq 0$
2) $(gL)/V^2 \ll 1$	Not used per se
3) $(V'/V) \approx -\delta C_{D_0}$ at high altitude ≈ 0 at low altitude	Not used per se
4) $f = 0$ for analytical purposes	$f \neq 0$
5) Steep, straight line trajectory	Any trajectory
6) Require $k \gg 1$ $[O(10^4)]$ for asymptotic solution for $k \not\gg 1$, exact solution is required.	We do not require $k \gg 1$ for our asymptotic solution. Actually $k = O(10^{1.7})$ in our application. We only require that coefficients vary slowly for our results.
7) Solutions are nonelementary (such as Bessel's and Kummer's)	Our solutions are in terms of elementary functions (such as sines, cosines, and algebraic functions).
8) Error analysis not possible in the general case.	Our theory leads to error theorems in the general case leading to strict and sharp bounds on the errors.

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